Source task and conventions

Prove the derivatives of a constant, a power, the exponential and the logarithm directly from the defining limits.

Conventional solution and proof

The derivative at x is $\lim_{h\to 0}[f(x+h)-f(x)]/h$, when this finite limit exists. For $f(x)=c$ the numerator is zero, so $f'(x)=0$. For an integer $n\geq 1$, the finite binomial identity gives

$$\frac{(x+h)^n-x^n}{h}=nx^{n-1}+\sum_{j=2}^{n}\binom{n}{j}x^{n-j}h^{j-1}.$$

The sum tends to zero; hence $(x^n)'=nx^{n-1}$. The case $n=0$ is constant. Define $e^x=\sum_{k=0}^{\infty}x^k/k!$. Absolute convergence permits the Cauchy product, and the binomial theorem gives $e^{x+h}=e^xe^h$. For $|h|\leq 1$,

$$\left|\frac{e^h-1}{h}-1\right|\leq |h|\sum_{k=2}^{\infty}\frac{1}{k!}\longrightarrow 0.$$

It follows that $(e^x)'=e^x$. The positive, continuous, strictly increasing exponential has a continuous inverse $\log:(0,\infty)\to\mathbb{R}$. Put $u=\log(x+h)-\log x$; then $u\to 0$ and $h=x(e^u-1)$. Consequently

$$\frac{\log(x+h)-\log x}{h}=\frac{1}{x}\frac{u}{e^u-1}\longrightarrow\frac{1}{x}.$$

For an integer $n=-m<0$, $x\neq 0$ and the algebraic identity