Source task and conventions

Derive the sum, product and quotient differentiation rules from first principles.

Conventional solution and proof

Suppose f, g are differentiable at x. Their increments satisfy $f(x+h) = f(x) + f'(x)h + r_f(h)$ and $g(x+h) = g(x) + g'(x)h + r_g(h)$, where $r_f(h)/h, r_g(h)/h \to 0$. Addition gives $(f+g)'(x) = f'(x) + g'(x)$. For the product, use the exact identity

$$f(x+h)g(x+h) - f(x)g(x) = [f(x+h) - f(x)]g(x+h) + f(x)[g(x+h) - g(x)].$$

Divide by h. Differentiability implies $g(x+h) \to g(x)$, so the limit is $f'(x)g(x) + f(x)g'(x)$. If $g(x) \neq 0$, continuity supplies a neighbourhood in which g is nonzero, and

$$\frac{1}{h}\left(\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{f(x+h)-f(x)}{h} - f(x)\frac{g(x+h)-g(x)}{h}}{g(x+h)g(x)}.$$

The denominator tends to $g(x)^2 \neq 0$, proving $(f/g)' = (f'g - fg')/g^2$.

Structural or analytic derivation

A first-order jet is the pair $(u, \dot{u})$ represented by $u + \varepsilon\dot{u}$ with $\varepsilon^2 = 0$. Multiplication in this two-dimensional algebra gives

$$(u + \varepsilon\dot{u})(v + \varepsilon\dot{v}) = uv + \varepsilon(\dot{u}v + u\dot{v}).$$