Source task and conventions

Explain $f'(x_0) = 0$ and give a function and point with this property.

Conventional solution and proof

The defining statement is

$$f(x_0 + h) - f(x_0) = o(h).$$

Thus the first-order change vanishes and the tangent line is horizontal. For $f(x) = x^2$ at $x_0 = 0$, the quotient $h^2/h = h \to 0$, and the point is a minimum. For $g(x) = x^3$ at zero, the quotient is $h^2 \to 0$, but g changes sign: a zero derivative alone does not imply an extremum. The function $-x^2$ supplies a maximum.

Structural or analytic derivation

A local power series $f(x_0 + h) = a_0 + a_1 h + a_2 h^2 + \cdots$ has $a_1 = 0$ at a stationary point. The first nonzero coefficient after a_0 determines the local behaviour: an even power gives a local extremum according to its sign, and an odd power changes sign. This conclusion follows by factoring $h^k(a_k + o(1))$, whose second factor has the sign of a_k for sufficiently small nonzero h.

Comparative cost and preferred method

The requested example and interpretation have fixed symbolic cost. The coefficient formulation additionally classifies analytic stationary points but supplies no computational acceleration for the single example. The conventional defining-limit answer is sufficient.