Source task and conventions

Find ∇f for $f(x_1, x_2) = 3x_1^2 + 5e^{x_2}$.

Conventional solution and proof

Holding the other coordinate fixed gives

$$\boxed{\nabla f(x_1, x_2) = (6x_1, 5e^{x_2})^{\mathsf{T}}.}$$

For an increment $h = (h_1, h_2)$, the residual after subtracting the linear term is $3h_1^2 + 5e^{x_2}(e^{h_2} - 1 - h_2) = O(\|h\|^2)$. Thus this vector is the total derivative, not merely a list of formal partial derivatives.

Structural or analytic derivation

The entire extension $F(z_1, z_2) = 3z_1^2 + 5e^{z_2}$ separates into two coordinate functions. Its first-order jet is $6z_1h_1 + 5e^{z_2}h_2$. Coefficient extraction or forward automatic differentiation therefore yields the same gradient without mixed terms.

Comparative cost and preferred method

The direct formula costs one exponential and two scalar multiplications. Automatic differentiation has constant overhead on the same expression graph. Separate finite differences use at least three function evaluations and introduce truncation error. Direct symbolic differentiation is the preferable method for this two-term expression.