Source task and conventions

For $u = f(x, y, z)$ and $x = x(a, b)$, $y = y(a, b)$, $z = z(a, b)$, give the derivatives of u with respect to a, b.

Conventional solution and proof

Assume $g = (x, y, z) : \mathbb{R}^2 \to \mathbb{R}^3$ and $f : \mathbb{R}^3 \to \mathbb{R}$ are differentiable at the relevant points. Write $g(t + h) = g(t) + J_g h + o(\|h\|)$ and $f(g(t) + k) = f(g(t)) + \nabla f^{\mathsf{T}} k + o(\|k\|)$. Since $k = O(\|h\|)$, substitution proves $D(f \circ g) = Df\, Dg$. Hence

$$u_a = f_x x_a + f_y y_a + f_z z_a, \qquad u_b = f_x x_b + f_y y_b + f_z z_b.$$

Every partial derivative of f is evaluated at $(x(a, b), y(a, b), z(a, b))$.

Structural or analytic derivation

The reverse accumulation is $\nabla_{(a,b)} u = J_g^{\mathsf{T}} \nabla f$. It propagates the three output sensitivities through one matrix–vector product. The displayed differential proof establishes the correctness of this accumulation rule. Complex variables require the corresponding real differential or a genuinely holomorphic map.

Comparative cost and preferred method

Once the five derivative objects are supplied, the matrix–vector product uses six multiplications and four additions. Computing the two displayed scalar sums has exactly the same count. Reverse-mode notation organizes the computation; it does not reduce these counts for the 3×2 example.