Source task and conventions

Find the derivative of a locally invertible differentiable function's inverse, specifying the necessary nondegeneracy condition.

Conventional solution and proof

Let f be continuous and one-to-one on an interval, differentiable at x_0, with $f'(x_0) \neq 0$. Its inverse g is continuous at $y_0 = f(x_0)$. For $y \neq y_0$, put $x = g(y)$; then $x \to x_0$ and

$$\frac{g(y) - g(y_0)}{y - y_0} = \left(\frac{f(x) - f(x_0)}{x - x_0} \right)^{-1} \longrightarrow \frac{1}{f'(x_0)}.$$

Therefore $g'(y) = 1/f'(g(y))$ wherever these hypotheses hold. Differentiating $f(g(y)) = y$ gives the same identity after existence of g' has been established.

Structural or analytic derivation

For a holomorphic F with $F'(z_0) \neq 0$, the complex inverse function theorem gives a local holomorphic inverse and the same formula. The sensitivity of the inverse is $|G'(F(z_0))| = 1/|F'(z_0)|$. If the derivative approaches zero, inversion becomes ill-conditioned. The real bijection $f(x) = x^3$ is a counterexample to differentiability of the inverse at zero.

Comparative cost and preferred method