Source task and conventions

Evaluate the derivative of $\log(x^2)\sin x + x^{-1}$ by both forward and reverse differentiation.

Conventional solution and proof

For forward mode set $\dot{x} = 1$. The tangents are

$$\dot{v}_1 = 2x, \quad \dot{v}_2 = 2/x, \quad \dot{v}_3 = \cos x,$$

$$\dot{v}_4 = (2/x)\sin x + \log(x^2)\cos x, \quad \dot{v}_5 = -x^{-2}, \quad \dot{v}_6 = \dot{v}_4 + \dot{v}_5.$$

For reverse mode start with $\bar{v}_6 = 1$. Then $\bar{v}_4 = \bar{v}_5 = 1$, $\bar{v}_2 = v_3$, $\bar{v}_3 = v_2$, and $\bar{v}_1 = \bar{v}_2/v_1$. Contributions to the single input accumulate as

$$\bar{x} = 2x\bar{v}_1 + \cos x\bar{v}_3 - x^{-2}\bar{v}_5 = \frac{2\sin x}{x} + \log(x^2)\cos x - x^{-2}.$$

Thus the two sweeps agree term by term. Every local rule is a proved scalar derivative, and applying the chain rule in topological or reversed topological order establishes correctness.