Source task and conventions

For independent Bernoulli(p) observations, compute the variance of the sample head fraction, a Chebyshev bound and the central-limit relation.

Conventional solution and proof

Let $X_i \in \{0,1\}$ with $\mathbb{P}(X_i = 1) = p$, independently, and $\widehat{p}_n = n^{-1}\sum_i X_i$. Since $X_i^2 = X_i$, $\mathbb{E}X_i = p$ and $\operatorname{Var} X_i = p - p^2$. Expanding the variance of the sum, every off-diagonal covariance vanishes by independence. Therefore

$$\mathbb{E}\widehat{p}_n = p, \qquad \boxed{\operatorname{Var}\widehat{p}_n = p(1-p)/n.}$$

For $\varepsilon > 0$, the pointwise inequality $\varepsilon^2 \mathbf{1}_{\{|\widehat{p}_n - p| \ge \varepsilon\}} \le (\widehat{p}_n - p)^2$ and expectation give

$$\mathbb{P}(|\widehat{p}_n - p| \ge \varepsilon) \le \frac{p(1-p)}{n\varepsilon^2} \le \frac{1}{4n\varepsilon^2}.$$

For $0 < p < 1$, set $Y = (X-p)/\sqrt{p(1-p)}$. Its characteristic function has expansion $\mathbb{E}e^{itY} = 1 - t^2/2 + O(|t|^3)$, because Y is bounded, has mean zero and variance one. Independence gives $[1 - t^2/(2n) + O(n^{-3/2})]^n \to e^{-t^2/2}$ for the normalized sum. By the characteristic-function continuity theorem, $\sqrt{n}(\widehat{p}_n - p)/\sqrt{p(1-p)}$ converges in distribution to the standard normal. At $p = 0$ or one the estimate is deterministic.

Structural or analytic derivation