Source task and conventions

Simplify the joint law of A, B, C when $A \to B \to C$ is a Markov chain.

Conventional solution and proof

The probability chain rule gives $P(a,b,c) = P(a)P(b \mid a)P(c \mid a,b)$ whenever the conditionals are defined. The Markov assumption is $C \perp A \mid B$, so $P(c \mid a,b) = P(c \mid b)$ on positive-probability conditioning events. Hence

$$\boxed{P(a,b,c) = P(a)P(b \mid a)P(c \mid b).}$$

On zero-probability conditioning events the joint mass is zero and conditional kernels may be defined arbitrarily without changing the joint law. The same factorization holds for densities when such densities exist.

Structural or analytic derivation

For finite state spaces, the first transition is a stochastic matrix K_{AB} and the second is K_{BC}. Marginalizing B gives the matrix composition $K_{AC} = K_{AB}K_{BC}$. The factorization is an exact finite-state representation. No unconditional independence of A and C is inferred.