## Source task and conventions

Count length-$n$ tilings with four square colours and five domino colours; give a recurrence, the first six positive-index terms and a closed form.

## Conventional solution and proof

Set $a_0 = 1$ for the empty tiling and $a_1 = 4$. For $n \geq 2$, the final tile is either one of four squares, leaving a length-$n-1$ tiling, or one of five dominoes, leaving a length-$n-2$ tiling. These cases are disjoint and exhaustive, so

$$a_n = 4a_{n-1} + 5a_{n-2}.$$

The requested terms are $4, 21, 104, 521, 2604, 13021$. The characteristic polynomial factors as $(r-5)(r+1)$. Thus $a_n = c5^n + d(-1)^n$ is a solution. The initial conditions give $c+d = 1$, $5c-d = 4$, hence $c = 5/6, d = 1/6$:

$$\boxed{a_n = \frac{5^{n+1} + (-1)^n}{6}.}$$

Substitution verifies the recurrence; equality at indices zero and one implies equality at every index by induction, proving uniqueness.

## Structural or analytic derivation

Summing the recurrence gives $(1 - 4z - 5z^2)A(z) = 1$. Since $1 - 4z - 5z^2 = (1-5z)(1+z)$,

$$A(z) = \frac{5/6}{1-5z} + \frac{1/6}{1+z}.$$