Source task and conventions

Solve $a_n = 4a_{n-1} - 4a_{n-2}$ in general and for initial pairs $(1, 2)$ and $(1, 8)$.

Conventional solution and proof

Set $b_n = a_n/2^n$. Division of the recurrence by 2^n gives $b_n = 2b_{n-1} - b_{n-2}$. Its first differences are constant, so

$$\boxed{a_n = \left[a_0 + n\left(\frac{a_1}{2} - a_0\right)\right] 2^n.}$$

Substitution verifies the recurrence and the two initial conditions, and induction proves uniqueness. The first pair gives $a_n = 2^n$; the second gives $a_n = (1 + 3n)2^n$. The repeated characteristic root is two, and the $n2^n$ term is necessary to span all initial-value solutions.

Structural or analytic derivation

Summation gives $A(z) = [a_0 + (a_1 - 4a_0)z]/(1 - 2z)^2$. Using $[z^n](1 - 2z)^{-1} = 2^n$ and $[z^n]z(1 - 2z)^{-2} = n2^{n-1}$ yields the same expression. The double pole accounts for the linear factor in n.

Comparative cost and preferred method

Rescaling and one first-difference invariant solve the recurrence directly. A remote exact term uses a binary shift for 2^n and arithmetic on $\Theta(n)$-bit integers; outputting those bits already costs $\Omega(n)$ bit operations. A full prefix takes linear recurrence updates. The analytic and characteristic-root methods are equivalent in asymptotic coefficient arithmetic.