Source task and conventions

Identify the sequences generated by the five rational functions in the question.

Conventional solution and proof

The geometric expansion and its derivative give:

$$\begin{array}{c|l} 4z/(1-z) & a_0 = 0,\ a_n = 4\ (n \geq 1) \\ 1/(1-4z) & a_n = 4^n \\ z/(1+z) & a_0 = 0,\ a_n = (-1)^{n-1}\ (n \geq 1) \\ 3z/(1+z)^2 & a_0 = 0,\ a_n = 3n(-1)^{n-1}\ (n \geq 1). \end{array}$$

For the last function, multiplying $1+z+z^2$ by $\sum_{n\geq 0}(n+1)z^n$ gives $a_0 = 1$, $a_1 = 3$, and $a_n = (n+1)+n+(n-1) = 3n$ for $n \geq 2$. Its sequence is $1, 3, 6, 9, 12, \ldots$. The fourth sequence is $0, 3, -6, 9, -12, \ldots$; the third is $0, 1, -1, 1, -1, \ldots$.

Structural or analytic derivation

All series are ordinary generating functions $A(z) = \sum_{n\geq 0} a_n z^n$. Identities hold formally: the coefficient of any fixed power uses finitely many operations. When complex-analytic evaluation is desired, the same identities hold inside the nearest pole. The geometric identity follows by multiplying $\sum_{n\geq 0} z^n$ by $1 - z$; coefficient cancellation leaves one.