Source task and conventions

Obtain the triangular-number generating function by two differentiations, differencing, and multiplication.

Conventional solution and proof

Use $T_n = n(n+1)/2$ for $n \geq 0$, so the indexed sequence is $0, 1, 3, 6, \ldots$. Let $G(z) = 1/(1-z)$. *Differentiation:* $zG'(z) + \frac{1}{2}z^2G''(z) = z/(1-z)^3$, and its coefficient is $n + n(n-1)/2 = T_n$. *Differencing:* $T_0 = 0$ and $T_n - T_{n-1} = n$, whence $(1-z)T(z) = \sum_{n\geq 1} nz^n = z/(1-z)^2$ and $T(z) = z/(1-z)^3$. *Multiplication:* the coefficient of z^n in $[z/(1-z)^2][1/(1-z)]$ is $\sum_{k=0}^{n} k = T_n$. Pairing k with $n-k$ proves this sum equals $n(n+1)/2$. If the first listed triangular number is one at index zero, the generating function is instead $T(z)/z = 1/(1-z)^3 = G''(z)/2$. Both indexing conventions are thereby explicit.

Structural or analytic derivation

The triple pole at one gives a degree-two coefficient polynomial. The identity $[z^n](1-z)^{-3} = \binom{n+2}{2}$ follows by multiplying three geometric series and counting nonnegative triples summing to n. Shifting by one gives $T_n = \binom{n+1}{2}$, agreeing with all three required derivations.

Comparative cost and preferred method

Each required derivation uses a constant number of rational operations. Directly summing the first n integers costs $n-1$ additions; the closed expression uses one product and division by two. The generating function and the conventional pairing proof both give the same efficient formula. The source explicitly asks for three derivations, so none is omitted to reduce page count.