Source task and conventions

Find the generating function for $a_n = 3a_{n-1} - a_{n-2}$, $a_0 = 1, a_1 = 5$.

Conventional solution and proof

Sum over $n \geq 2$: $A - 1 - 5z = 3z(A - 1) - z^2 A$. Rearrangement yields

$$\boxed{A(z) = \frac{1 + 2z}{1 - 3z + z^2}.}$$

The coefficient at zero is one, at one is five, and the coefficients from index two satisfy the source recurrence because multiplication by the denominator cancels them. This proves both correctness and uniqueness.

Structural or analytic derivation

The two roots of $r^2 - 3r + 1$ are $r_\pm = (3 \pm \sqrt{5})/2$. Partial fractions give

$$a_n = \frac{7 + \sqrt{5}}{2\sqrt{5}} r_+^n + \frac{\sqrt{5} - 7}{2\sqrt{5}} r_-^n.$$

The constants solve $c_+ + c_- = 1$ and $c_+ r_+ + c_- r_- = 5$. This optional coefficient formula shows how the poles encode the two recurrence modes.

Comparative cost and preferred method