Source task and conventions

Use the Fibonacci recurrence to derive its generating function.

Conventional solution and proof

Use the source's $F_0 = 0, F_1 = 1$ convention. For $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$. Writing $F(z) = \sum_{n \geq 0} F_n z^n$ gives $F - z = zF + z^2F$, and therefore

$$\boxed{F(z) = z/(1 - z - z^2).}$$

Multiplication by the denominator recovers the initial terms and recurrence, proving the result. For the shifted sequence $1, 1, 2, 3, \ldots$ beginning at zero, the function is $F(z)/z = 1/(1 - z - z^2)$; this shift is used explicitly in Exercise 13.

Structural or analytic derivation

The factorization $1-z-z^2 = (1-\varphi z)(1-\psi z)$, with $\varphi = (1+\sqrt{5})/2$, $\psi = (1-\sqrt{5})/2$, yields $F_n = (\varphi^n - \psi^n)/\sqrt{5}$. Substitution of the two roots and the initial conditions is an independent proof of this coefficient formula.

Comparative cost and preferred method

The fixed-order recurrence and output-size comparison in AC-035 applies to this coefficient sequence. The calculation above supplies its particular initial data and rational function.