Source task and conventions

Starting with $(1-z)^{-2}$, obtain the four generating functions in the source.

Conventional solution and proof

Let $G(z)=\sum_{n\ge 0}(n+1)z^n=(1-z)^{-2}$. Then

	generating function	coefficient rule
(a)	$G(z^2)=(1-z^2)^{-2}$	$a_{2k}=k+1,\ a_{2k+1}=0$
(b)	$G(-z)=(1+z)^{-2}$	$a_n=(-1)^n(n+1)$
(c)	$3zG(z)=3z/(1-z)^2$	$a_n=3n$
(d)	$3zG(z)/(1-z)=3z/(1-z)^3$	$a_n=3n(n+1)/2.$

The fourth is the cumulative sum of the third: $0,3,9,18,30,\ldots$. The shift and substitution rules follow by replacing powers in the defining series, and the cumulative-sum rule follows by the finite coefficient Cauchy product.

Structural or analytic derivation

The formal-series convention and geometric-series justification are established in AC-036. The displayed transformation uses that same coefficientwise arithmetic on its stated convergence domain.