Source task and conventions

For $A(z)$ generating $3, 5, 9, 15, 23, 33, \ldots$, find the generating function of consecutive differences in terms of A, identify it directly, and solve for A.

Conventional solution and proof

Let $b_n = a_{n+1} - a_n$, so $b_0 = 2, b_1 = 4, b_2 = 6, \ldots$. Then

$$B(z) = \sum_{n \geq 0} (a_{n+1} - a_n) z^n = \frac{A(z) - 3}{z} - A(z).$$

Since $b_n = 2(n+1)$, $B(z) = 2/(1-z)^2$. Solving $zB = (1-z)A - 3$ gives

$$\boxed{A(z) = \frac{3}{1-z} + \frac{2z}{(1-z)^3} = \frac{3 - 4z + 3z^2}{(1-z)^3}.}$$

The coefficient formula is $a_n = 3 + n(n+1)$; subtracting consecutive values gives $2(n+1)$ and $a_0 = 3$, proving the reconstruction.

Structural or analytic derivation

Forward differencing removes a pole order from polynomial sequences. Here an affine first difference gives a quadratic original coefficient, represented by a triple pole at one. The initial value supplies the independent constant-sequence term.