Source task and conventions

Solve all parts of CLRS Problem 4-5 using $F_0 = 0, F_1 = 1$: generating-function identity, factorization, coefficient formula, nearest-integer statement and the lower bound $F_{i+2} \geq \varphi^i$.

Conventional solution and proof

(a) Multiply $F_n = F_{n-1} + F_{n-2}$ by z^n and sum for $n \geq 2$. The initial terms give $F(z) = z + zF(z) + z^2F(z)$. (b) Therefore $F(z) = z/(1 - z - z^2)$. Let $\varphi = (1 + \sqrt{5})/2$ and $\psi = (1 - \sqrt{5})/2$. Their sum is one, product is minus one, and difference is $\sqrt{5}$, so

$$F(z) = \frac{z}{(1 - \varphi z)(1 - \psi z)} = \frac{1}{\sqrt{5}} \left(\frac{1}{1 - \varphi z} - \frac{1}{1 - \psi z} \right).$$

(c) Expanding the two formal geometric series yields $F_n = (\varphi^n - \psi^n)/\sqrt{5}$. Formal coefficient uniqueness justifies the identification without imposing a numerical convergence domain. (d) Since $|\psi| < 1$, $|F_i - \varphi^i/\sqrt{5}| = |\psi|^i/\sqrt{5} < 1/2$. The integer F_i is therefore the unique nearest integer for every $i > 0$. (e) For $i = 0$, $F_2 = 1 = \varphi^0$; for $i = 1$, $F_3 = 2 \geq \varphi$. If the inequality holds at $i - 1, i - 2$, then $F_{i+2} = F_{i+1} + F_i \geq \varphi^{i-1} + \varphi^{i-2} = \varphi^i$, since $\varphi^2 = \varphi + 1$. Induction completes the proof.

Structural or analytic derivation

The resolvent denominator has two simple poles and coefficient extraction resolves the recurrence into its two spectral modes. For exact computation, avoid rounding an insufficiently precise floating Binet value. Fast doubling uses

$$F_{2k} = F_k(2F_{k+1} - F_k), \qquad F_{2k+1} = F_k^2 + F_{k+1}^2.$$