Source task and conventions

Solve CLRS Problem 12-4: the tree recurrence, quadratic generating function, exact Catalan coefficient and asymptotic estimate.

Conventional solution and proof

(a) The empty tree is unique, so $b_0 = 1$. A nonempty rooted ordered binary tree has one root, k left-subtree nodes and $n-1-k$ right-subtree nodes. This unique decomposition gives $b_n = \sum_{k=0}^{n-1} b_k b_{n-1-k}$. (b) The coefficient Cauchy product yields $B = 1 + zB^2$. Solving the quadratic and choosing the branch whose constant coefficient is one gives

$$B(z) = \frac{1 - \sqrt{1-4z}}{2z};$$

the other sign has a negative-power term and is not the required power series. (c) The formal binomial expansion gives

$$b_n = -\frac{1}{2}\binom{1/2}{n+1}(-4)^{n+1} = \frac{(2n)!}{n!(n+1)!} = \frac{1}{n+1}\binom{2n}{n}.$$

For the algebraic simplification, expand the numerator of the generalized binomial coefficient as $(1/2)(-1/2)(-3/2)\cdots(-(2n-1)/2)$ and use $(2n)! = 2^n n!(1 \cdot 3 \cdots (2n-1))$. The case $n = 0$ is one directly. The recurrence and initial value determine every coefficient uniquely, so the formula counts the original trees. (d) Stirling's formula with remainder, $n! = \sqrt{2\pi n}(n/e)^n e^{r_n}$ with $r_n = O(1/n)$, gives $\binom{2n}{n} = 4^n(\pi n)^{-1/2}(1 + O(1/n))$. Dividing by $n+1 = n(1+1/n)$ proves

$$\boxed{b_n = \frac{4^n}{\sqrt{\pi}\, n^{3/2}}(1 + O(1/n)).}$$

The displayed error follows because $r_{2n} - 2r_n = O(1/n)$ and $e^{O(1/n)} = 1 + O(1/n)$.