Source task and conventions

Compute the quotient and remainder of $A(x)$ divided by $x - x_0$ in $\Theta(n)$ time, where $\deg A < n$.

Conventional solution and proof

For $n = 1$ the quotient is zero and the remainder is a_0. For $n \geq 2$, define

$$q_{n-2} = a_{n-1}, \qquad q_{j-1} = a_j + x_0 q_j \quad (j = n-2, \ldots, 1), \qquad r = a_0 + x_0 q_0.$$

Let $Q(x) = \sum_{j=0}^{n-2} q_j x^j$. The leading coefficient of $(x - x_0)Q + r$ is $q_{n-2} = a_{n-1}$; each middle coefficient is $q_{j-1} - x_0 q_j = a_j$; and the constant is $r - x_0 q_0 = a_0$. Thus $A = (x - x_0)Q + r$ identically, and substitution gives $r = A(x_0)$.

Structural or analytic derivation

The descending recurrence is Horner's rule with its intermediate values retained. It is also repeated extraction of the constant term in powers of $x - x_0$. The quotient coefficients are obtained during the same pass that evaluates the remainder; no separate interpolation is needed.

Comparative cost and preferred method

There are $n - 1$ multiplications and $n - 1$ additions after copying the leading coefficient, with $n - 1$ quotient outputs and one remainder. The $\Omega(n)$ input/output lower bound proves $\Theta(n)$ optimality for this representation. The standard synthetic-division algorithm is retained; a transform adds unnecessary setup.