Source task and conventions

Use the Lagrange formula to interpolate n distinct point-value pairs in $\Theta(n^2)$ arithmetic.

Conventional solution and proof

Construct $P(x) = \prod_{j=0}^{n-1}(x - x_j)$ by multiplying successive linear factors, using $O(n^2)$ coefficient arithmetic. For each k, synthetic division yields $P_k = P/(x - x_k)$ in $O(n)$ time. Compute $d_k = \prod_{j \neq k}(x_k - x_j)$ in $O(n)$ time; it is nonzero because the nodes are distinct. Accumulate

$$\boxed{A(x) = \sum_{k=0}^{n-1} \frac{y_k}{d_k} P_k(x).}$$

At $x = x_i$ every summand vanishes except $k = i$, whose value is y_i. The result has degree below n, so the preceding interpolation-uniqueness proof establishes correctness. Building P, all synthetic divisions, all denominators and all coefficient accumulations each use at most $O(n^2)$ operations.

Structural or analytic derivation

The quotient $P/(x - x_k)$ is the polynomial numerator of a residue-like interpolation basis, and $d_k = P'(x_k)$. A balanced product/remainder tree can eventually reduce large problems to quasi-linear polynomial arithmetic, but the source specifically asks for the elementary quadratic construction. The displayed algorithm supplies it completely.

Comparative cost and preferred method