Conventional solution and proof

Let $A = BQ + R$ with $\deg R < \deg B$. At any node where $B(x_i) \neq 0$,

$$A(x_i)/B(x_i) = Q(x_i) + R(x_i)/B(x_i).$$

If $R = 0$, the ratios are genuine quotient values. They determine Q when enough distinct nonzero-denominator nodes are retained and a suitable quotient degree bound is known. However, a root of B is also a root of A, so the ratio at such a node is $0/0$ and is undefined; for example $A = x^2, B = x$ at zero. The degree bound of Q can also be lower than the original representation's bound. If $R \neq 0$, the ratios are values of a rational function, not generally values of the polynomial quotient. For $A = x^2$, $B = x + 1$, one has $Q = x - 1, R = 1$. Ratios at zero and one are zero and $1/2$, whose linear interpolant is $x/2$, not $x - 1$. Thus the naive method fails even with no zero denominators.

Structural or analytic derivation

Polynomial division is a decomposition into a polynomial part at infinity and a proper rational remainder. Pointwise evaluation of the rational function does not separate these two parts. An analytic Laurent expansion at infinity would identify Q, but it must be accompanied by the degree and pole information that the naive division discards.

Comparative cost and preferred method

Exact division from valid values can be useful with an established quotient bound. General Euclidean division or fast polynomial division supplies the correct output directly. The counterexample uses two evaluations and completely refutes universal correctness; no benchmark can turn the invalid ratio rule into a valid faster method.