Source task and conventions

Generalize the FFT to $n = 3^s$, give its recurrence and solve the recurrence.

Conventional solution and proof

Let $m = n/3$ and split a_j by $j = 3r + q$, $q = 0, 1, 2$. Compute the three length-m transforms $B_q(k) = \sum_{r=0}^{m-1} a_{3r+q}\omega_m^{rk}$. For $k = 0, \ldots, m-1$ and $t = 0, 1, 2$, combine

$$\boxed{y_{k+tm} = B_0(k) + \omega_n^{k+tm} B_1(k) + \omega_n^{2(k+tm)} B_2(k).}$$

Substitution of $j = 3r + q$ in the original DFT gives the formula because $\omega_n^{3r(k+tm)} = \omega_m^{rk}$. The base transform of one entry returns that entry. This proves the recursive algorithm by induction on s.

Structural or analytic derivation

The output consists of three cosets of the smaller frequency lattice. The local three-point Fourier matrix combines them, with ω_3^t twiddle factors. Precomputing successive powers of ω_n by multiplication keeps the combine stage linear.

Comparative cost and preferred method

There are three subproblems of size $n/3$ and a cn combine cost. Thus $T(n) = 3T(n/3) + cn$, $T(1) = c_0$. Every one of the $\log_3 n$ nontrivial levels costs cn, so $T(n) = c_0 n + cn \log_3 n = \Theta(n \log n)$. The source's generalized FFT is already the preferable method for this transform family.