Conventional solution and proof

For $z \neq 0$, choose either square root q with $q^2 = z$. The identity $k^2 + j^2 - (k-j)^2 = 2kj$ gives

$$y_k = q^{k^2} \sum_{j=0}^{n-1} (a_j q^{j^2}) q^{-(k-j)^2}.$$

Set $b_j = a_j q^{j^2}$ and $c_l = q^{-l^2}$ for $-(n-1) \leq l \leq n-1$. Choose a power-of-two $M \geq 2n-1$, zero-pad b to M, and store c_l at index $l \bmod M$. These indices are distinct on the required difference interval. An M-point circular convolution then supplies the indicated sum for $k = 0, \ldots, n-1$, since every difference $k-j$ lies in that interval. Multiply by q^{k^2} to finish. Changing the sign of q does not change the result because the combined exponent is even. For $z = 0$, interpret $z^0 = 1$ as in the polynomial definition. Then $y_0 = \sum_j a_j$ and $y_k = a_0$ for $k \geq 1$. This case costs $O(n)$ and avoids the invalid negative powers in the chirp formula.

Structural or analytic derivation

The phase is a quadratic form whose polarization converts the bilinear term kj into a difference convolution. All chirp powers can be prepared in $O(n)$ multiplications: starting from $q^0 = 1$, update $q^{(j+1)^2} = q^{j^2} q^{2j+1}$ and update the ratio by q^2. The inverse-power sequence is generated analogously. This is a complete Bluestein-type construction for arbitrary nonzero parameter, not only roots of unity.

Comparative cost and preferred method