Source task and conventions

Solve every part of CLRS Problem 30-3 for a degree-below-n polynomial and a point x_0.

Conventional solution and proof

(a) If $A(x) = \sum_{j=0}^{n-1} b_j (x - x_0)^j$, differentiation gives $A^{(t)}(x_0) = t! b_t$. Successively update $t!$ and multiply each b_t, using $O(n)$ arithmetic. (b) The values $A(x_0 + \omega_n^k)$ are the DFT of b. One inverse FFT therefore recovers all b_t in $O(n \log n)$. (c) Expand each monomial by the binomial theorem:

$$A(x_0 + w) = \sum_{j=0}^{n-1} a_j \sum_{r=0}^{j} \binom{j}{r} x_0^{j-r} w^r = \sum_{r=0}^{n-1} \frac{w^r}{r!} \sum_{j=r}^{n-1} a_j j! \frac{x_0^{j-r}}{(j-r)!}.$$

Set $f(j) = a_j j!$ and $g(l) = x_0^{-l}/(-l)!$ for $-(n-1) \le l \le 0$, with $g(l) = 0$ for $l > 0$. At $w = \omega_n^k$ this is exactly the source's asserted convolution formula, with $g(r - j)$ zero for $j < r$. (d) Form $u_l = a_{n-1-l}(n - 1 - l)!$ and $v_l = x_0^l / l!$ for $0 \le l < n$. Their linear convolution satisfies

$$(u * v)_{n-1-r} = \sum_{j=r}^{n-1} a_j j! x_0^{j-r}/(j-r)! = A^{(r)}(x_0).$$

One zero-padded FFT convolution thus gives all derivatives directly. Dividing them by $r!$ and taking one FFT supplies all shifted evaluations, also within $O(n \log n)$. The convention $x_0^0 = 1$ handles $x_0 = 0$ without dividing by x_0.