Source task and conventions

Given n roots with multiplicities, compute coefficients of a degree-n polynomial with exactly those roots in $O(n \log^2 n)$ arithmetic.

Conventional solution and proof

Choose the monic polynomial $P(x) = \prod_{j=0}^{n-1}(x - z_j)$. Every listed value is a zero with its specified multiplicity, and over $\mathbb{C}$ a nonzero product vanishes only when one of its factors does, so there are no other zeros. Build a balanced product tree: leaves are the n linear factors; each internal node multiplies its two child polynomials by an FFT polynomial multiplication. Pad missing leaves with the constant polynomial one to reach a power of two. Induction up the tree proves that every node equals the product of its leaf factors, so the root is the desired P.

Structural or analytic derivation

At level l, each product has degree $O(2^l)$ and there are $O(n/2^l)$ products. With multiplication cost $M(d) = O(d \log d)$, the work at that level is $O(nl)$. Summing $l = 1, \ldots, \lceil \log_2 n \rceil$ gives $O(n \log^2 n)$. This explicitly proves the requested bound, including repeated roots.

Comparative cost and preferred method

Sequential multiplication by linear factors takes $\Theta(n^2)$ coefficient operations. The balanced tree has the proved $O(n \log^2 n)$ arithmetic cost; breadth-first construction can retain only adjacent levels with $O(n)$ coefficients. Repeated or nearby roots can make coefficient/root conversion numerically sensitive, so algebraic correctness and floating stability are separate.