Source task and conventions

Prove $\det V(x_0, \ldots, x_{n-1}) = \prod_{j<k}(x_k - x_j)$, where row i is $(1, x_i, \ldots, x_i^{n-1})$.

Conventional solution and proof

For $n = 1$ both sides equal one. For $n > 1$, working right to left for $j = n-1, \ldots, 1$, replace column j by column $j - x_0$ times column $j-1$; in conventional notation the operation is $C_j \leftarrow C_j - x_0 C_{j-1}$. Because the order is right to left, the subtracted column has not yet changed. These determinant-preserving operations make the first row $(1, 0, \ldots, 0)$. In row $i \geq 1$, column $j \geq 1$ is $x_i^j - x_0 x_i^{j-1} = (x_i - x_0)x_i^{j-1}$. Expand along the first row and factor $x_i - x_0$ from each remaining row. The residual determinant is $V(x_1, \ldots, x_{n-1})$. Induction gives

$$\det V = \prod_{i=1}^{n-1}(x_i - x_0) \prod_{1 \leq j < k < n}(x_k - x_j) = \prod_{0 \leq j < k < n}(x_k - x_j).$$

The proof remains valid when nodes coincide, where both sides vanish.