Source task and conventions

Derive the Newton update, establish the correct local error relation and iteration bound, and compare Newton with gradient descent for $(x-3)^2$ from $x_0 = 3.5$.

Conventional solution and proof

(a) Setting the tangent-line equation to zero gives $x_{t+1} = x_t - f(x_t)/f'(x_t)$ whenever $f'(x_t) \neq 0$. (b) The printed Taylor and error formulas contain defects. Taylor's theorem with Lagrange remainder is

$$0 = f(x_*) = f(x_t) + f'(x_t)(x_* - x_t) + \tfrac{1}{2} f''(\xi_t)(x_* - x_t)^2.$$

Subtracting the Newton update gives the exact relation

$$\boxed{e_{t+1} = \frac{|f''(\xi_t)|}{2|f'(x_t)|}\, e_t^2, \qquad e_t = |x_t - x_*|.}$$

The derivative in the denominator is at x_t, and the error is squared. The displayed source formula with a first power of e_t is not the quadratic-convergence identity. (c) Choose a closed interval about x_* lying in the differentiability domain, with $|f'| \geq m > 0$ and $|f''| \leq M$, and set $C = M/(2m)$. If $Ce_0 < 1$ and the initial error ball lies in this interval, then $e_{t+1} \leq Ce_t^2 < e_t$, so every iterate remains inside the interval. Induction gives

$$e_t \leq C^{-1}(Ce_0)^{2^t}.$$

For $C > 0$ and a target smaller than the initial error, it suffices that

$$t \geq \left\lceil \log_2 \left(\frac{\log(1/(C\delta))}{\log(1/(Ce_0))} \right) \right\rceil,$$