## Source task and conventions

Give a proved $O(n^2)$ algorithm for an optimal bitonic tour when all point $x$ coordinates are distinct.

## Conventional solution and proof

Sort the points as $p_1, \ldots, p_n$ by increasing $x$. Let $d(i, j)$ be Euclidean distance. For $i < j$, let $B[i, j]$ be the minimum total length of two increasing paths starting at $p_1$, covering exactly $p_1, \ldots, p_j$, sharing only $p_1$, and ending at $p_i, p_j$. The base is $B[1, 2] = d(1, 2)$. For $j \geq 3$,

$$B[i, j] = B[i, j-1] + d(j-1, j) \quad (i < j-1),$$

$$B[j-1, j] = \min_{k<j-1} \{B[k, j-1] + d(k, j)\}.$$

For the first case, the new rightmost point must extend the path whose endpoint was $j-1$; removing its last edge gives exactly the preceding state. In the second case it extends the other path, whose previous endpoint is some $k < j-1$; removing and then restoring that edge gives all and only the alternatives in the minimum. Induction on $j$ therefore proves optimality of every state. The optimal closed length is $B[n-1, n] + d(n-1, n)$. Equivalently, minimize $B[i, n] + d(i, n)$ over $i < n$ and use the recurrence to obtain the same value. Store each minimizing $k$. Backtracking the recurrence adds the recorded edge and recursively visits the previous state; append the final closing edge to recover the tour. For two points the closed tour traverses the segment twice; one point has length zero.

## Structural or analytic derivation