Source task and conventions

Construct DFAs for exactly three b symbols, a multiple-of-three b count, and a difference of a and b counts divisible by three.

Conventional solution and proof

For (a), use states $0, 1, 2, 3, \perp$, start zero and accept only three. An a leaves the state unchanged. A b sends $0 \to 1 \to 2 \to 3 \to \perp$, and $\perp$ loops. Induction on prefix length shows that states zero through three equal the b count and $\perp$ means at least four. For (b), use states $0, 1, 2$, start/accept zero, with $\delta(r, a) = r$ and $\delta(r, b) = r + 1 \pmod 3$. The invariant is the b count modulo three. For (c), use the same start and accepting state with $\delta(r, a) = r + 1 \pmod 3$ and $\delta(r, b) = r - 1 \pmod 3$. Its invariant is $\#a - \#b \pmod 3$. Each invariant begins correctly at the empty word and is preserved by the specified transition, proving all three languages.

Structural or analytic derivation

The modular DFAs are actions of the cyclic group $\mathbb{Z}/3\mathbb{Z}$. Their transition matrices are permutation matrices; a roots-of-unity diagonalization can count accepted words of a fixed length. The group invariant itself is already the simplest recognition proof. For exact-three counting, a truncated nonnegative counter with overflow is the appropriate finite-state object.

Comparative cost and preferred method