Conventional solution and proof

For $S = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$, the normalized eigenvectors are $u_1 = (1,1)^\top/\sqrt{2}$ with eigenvalue 3, and $u_2 = (1,-1)^\top/\sqrt{2}$ with eigenvalue 1. Hence

$$Q = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}, \quad S = Q\Lambda Q^\top.$$

One checks $Q^\top Q = I$ and the product explicitly. A decomposition is not literally unique: columns may be sign-changed or permuted together with the diagonal. Ordering the eigenvalues decreasingly and choosing positive first coordinates gives the displayed convention. The preceding source prose lists the two eigenvectors in the reverse correspondence to its listed eigenvalues; direct multiplication fixes the pairing.

Structural or analytic derivation

The two orthogonal directions are the symmetric and antisymmetric combinations of coordinates. Their invariant one-dimensional subspaces diagonalize the operator without determinant expansion.

Comparative cost and preferred method

Two additions/subtractions provide the invariant directions. The symmetry method is simpler for this special matrix; it is not a general complexity result for arbitrary symmetric matrices.