Source task and conventions

Prove the equality in Exercise 18.9 with dimensions made explicit.

Conventional solution and proof

Here $U_1 = u_1 \in \mathbb{R}^{M\times 1}$, $V_1 = v_1 \in \mathbb{R}^{N\times 1}$ and σ_1 is a scalar. For every entry (i, j),

$$(U_1\sigma_1 V_1^\top)_{ij} = u_{i1}\sigma_1 v_{j1} = \sigma_1(u_1 v_1^\top)_{ij}.$$

Thus $U_1\sigma_1 V_1^\top = \sigma_1 u_1 v_1^\top$. Its image is contained in span u_1, so it has rank at most one and exactly one when $\sigma_1 > 0$ and the vectors are nonzero. For any vector x, its action is $\sigma_1 u_1(v_1^\top x)$.

Structural or analytic derivation

The outer-product form is an exact separable representation of the two-index matrix. It is the finite-dimensional prototype of a rank-one operator on a Hilbert space.

Comparative cost and preferred method

Applying the factors costs N dot-product terms and M scalar multiplications, versus MN products for a materialized dense matrix. Producing all MN entries still costs $\Omega(MN)$ output work. The benefit applies to operator actions and storage, not to avoiding a requested full output.