Source task and conventions

Prove parts (a)–(k). In part (e), identify the domain issue at zero.

Conventional solution and proof

We use three elementary closure facts. A feature inner product is PDS because its quadratic form is $\|\sum_i c_i\phi(x_i)\|^2$. Sums and nonnegative integrals of PDS kernels are PDS by linearity of finite quadratic forms. Products are PDS: for finite Gram matrices write $G_{ij} = \sum_a v_{ia}v_{ja}$ and $H_{ij} = \sum_b w_{ib}w_{jb}$; then $(G \circ H)_{ij} = \sum_{a,b}(v_{ia}w_{ib})(v_{ja}w_{jb})$. Nonnegative power series follow by finite partial sums and pointwise limits. In particular,

$$e^{-t\|x-y\|^2} = e^{-t\|x\|^2}e^{-t\|y\|^2}\sum_{n=0}^{\infty}\frac{(2t)^n}{n!}\langle x, y\rangle^n$$

is PDS for $t \geq 0$, since tensor powers represent the inner-product powers.

(a) $\cos(x-y) = \cos x\cos y + \sin x\sin y$, with feature $(\cos x, \sin x)$. (b) Compose this feature with $x \mapsto x^2$. (c) Each summand is the inner product of the nth tensor powers of $(\cos x_i^2, \sin x_i^2)$; their direct sum is a feature for the displayed sum.

(d) For $x, y > 0$, $(x+y)^{-1} = \int_0^\infty e^{-tx}e^{-ty}\,dt$, the inner product of the functions e^{-tx} in $L^2(0,\infty)$. They have squared norm $1/(2x) < \infty$.

(e) For nonzero vectors, $K(x,y) = \langle x/\|x\|, y/\|y\|\rangle$. The angle with a zero vector is undefined, so the source expression does not define a kernel on all of $\mathbb{R}^n$. The stated proof applies on $\mathbb{R}^n \setminus \{0\}$. Defining the feature at zero to be zero provides a PDS extension, with $K(0,y) = 0$.

(f) Set $v(x) = (\cos 2x, \sin 2x)$. Since $\|v(x)-v(y)\|^2 = 4\sin^2(x-y)$, the kernel is the Gaussian $e^{-(\lambda/4)\|v(x)-v(y)\|^2}$.