(i) The binomial expansion is

$$(1-s)^{-1/2} = \sum_{n\geq 0} b_n s^n, \qquad b_n = \binom{2n}{n}/4^n > 0.$$

For $\|x\|, \|y\| < 1$, use the Hilbert direct-sum feature $\phi(x) = (\sqrt{b_n}\, x^{\otimes n})_{n\geq 0}$. Its norm is finite because $\sum b_n \|x\|^{2n} = (1-\|x\|^2)^{-1/2}$. Its inner product is the required kernel.

(j) Since $(1+s)^{-1} = \int_0^\infty e^{-u} e^{-us}\, du$, insert $s = \|x-y\|^2/\sigma^2$ and use Gaussian positivity.

(k) The base kernel $J(x,y) = \sum_i \min(|x_i|, |y_i|)$ is PDS through the direct-sum features $\mathbf{1}_{[0,|x_i|]}(t)$. Therefore $e^{J/\sigma^2} = \sum_{n\geq 0} J^n/(\sigma^{2n} n!)$ is PDS. All series and integrals used above converge at every pair in the declared domain, so finite quadratic forms pass to their limits.

Structural or analytic derivation

The certificates specify finite features in (a), (b), and the tensor construction in (c), square-integrable functions in (d), (h), and (k), and positive mixtures or analytic series in (f), (g), (i), and (j). These representations also expose rank and singularities: (a)–(b) have sample Gram rank at most two; (i) requires strict unit-ball membership; (e) requires a nonzero norm or an explicit extension.

Comparative cost and preferred method

A numerical eigenvalue test on one m-point sample cannot prove the all-samples definition. The feature/integral proofs do. For (a)–(b), a Gram action Gc can use a two-column feature matrix in $O(m)$ operations and $O(m)$ storage, compared with $O(m^2)$ for a materialized dense Gram matrix. For the remaining kernels, positivity alone supplies no finite-rank reduction or approximation accuracy. The preferred method for the proof is structural; the preferred evaluation algorithm depends on the kernel and data.