Source task and conventions

Interpret the source hint W and prove the resulting graph kernel is PDS.

Conventional solution and proof

Take the symmetric weighted adjacency matrix W, with zero for absent edges. The sum over length-two walks from p to q is

$$K(p,q) = \sum_{r \in V} W_{pr} W_{rq} = (W^2)_{pq}.$$

For any real vector c, $c^\top K c = c^\top W^\top W c = \|Wc\|^2 \geq 0$. No nonnegativity assumption on the edge weights is necessary. Symmetry is immediate.

Structural or analytic derivation

The feature of vertex p is its weighted adjacency row. The inner product of two such features sums their weighted common-neighbour contributions. If "path" is required to be simple and hence to exclude the walk p, r, p, the claimed theorem fails: a three-vertex path has zero diagonal and a positive $(1,3)$ entry, giving a