Source task and conventions

Prove positivity of $\exp(-|A \triangle B|/2)$.

Conventional solution and proof

Enumerate the finite universe and associate each set A with its indicator vector v_A. Each coordinate contributes one to $\|v_A - v_B\|^2$ precisely when membership differs, so $|A \triangle B| = \|v_A - v_B\|^2$. The Gaussian expansion proved in the preceding kernel dossier shows the kernel is PDS. Explicitly,

$$K(A, B) = e^{-|A|/2} e^{-|B|/2} e^{|A \cap B|},$$

and $|A \cap B| = v_A^\top v_B$ is a feature inner product; its exponential is a convergent nonnegative sum of tensor-power kernels.

Structural or analytic derivation

A second proof factors over the universe: $K(A, B) = \prod_j k(v_{A,j}, v_{B,j})$, where the two-state Gram matrix of k is $\begin{pmatrix} 1 & e^{-1/2} \\ e^{-1/2} & 1 \end{pmatrix}$. Its eigenvalues $1 \pm e^{-1/2}$ are positive. Tensor products preserve positivity, including all $2^{|X|}$ sets simultaneously.

Comparative cost and preferred method

The product proof reduces positivity on an exponentially large set domain to a 2×2 calculation and a proved closure rule. Kernel evaluation can use a bitwise symmetric difference and population count; this conventional implementation already realizes the set identity efficiently. No additional complex-analytic speedup is asserted.