Source task and conventions

Prove parts (a) and (b) under the NDS zero-sum convention.

Conventional solution and proof

For a symmetric kernel, NDS means $\sum_{i,j} c_i c_j K(x_i, x_j) \le 0$ whenever $\sum_i c_i = 0$. (a) $\sin^2(x-y) = (1 - \cos(2x-2y))/2$. The constant term has zero quadratic form, while the cosine term has quadratic form $(\sum_i c_i \cos 2x_i)^2 + (\sum_i c_i \sin 2x_i)^2$. Its negative half is nonpositive. (b) For $a > 0$,

$$\log a = \int_0^\infty \frac{e^{-t} - e^{-at}}{t}\, dt.$$

The integral converges at zero and infinity; its derivative in a is $\int_0^\infty e^{-at} dt = 1/a$ and it vanishes at $a = 1$, proving the identity. Inserting $a = x_i + x_j$ and using $\sum c_i = 0$ gives

$$\sum_{i,j} c_i c_j \log(x_i + x_j) = -\int_0^\infty \frac{(\sum_i c_i e^{-t x_i})^2}{t}\, dt \le 0.$$

For positive x_i, decay at infinity is exponential; at zero the numerator is $O(t^2)$ because the coefficients sum to zero. Thus the displayed integral and the finite interchange are justified.

Structural or analytic derivation

The negative quadratic forms are explicitly squared norms. Part (a) uses a two-dimensional trigonometric feature; part (b) uses a Laplace feature with the constant component eliminated by the zero-sum constraint.

Comparative cost and preferred method