Source task and conventions

Determine both signs for $K(x,y) = \|x-y\|^{3/2}$.

Conventional solution and proof

The two points $0, e_1$ give the same indefinite matrix as the preceding question, so K is not PDS. For NDS, put $\alpha = 3/4$. Integration by parts, with vanishing endpoint terms, yields for $s \geq 0$

$$s^\alpha = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty (1-e^{-ts}) t^{-1-\alpha}\, dt.$$

Indeed the substitution $u = ts$ reduces the integral to $s^\alpha \Gamma(1-\alpha)/\alpha$; at zero the integrand is $O(t^{-\alpha})$ and at infinity $O(t^{-1-\alpha})$. For coefficients summing to zero,

$$\sum_{i,j} c_i c_j \|x_i - x_j\|^{3/2} = -\frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty c^\top G_t c\; t^{-1-\alpha}\, dt \leq 0,$$

where $(G_t)_{ij} = e^{-t\|x_i - x_j\|^2}$ is PDS by the Gaussian feature expansion. The finite sum of convergent difference integrals justifies the interchange.

Structural or analytic derivation

The exponent $3/2$ is twice a number in $(0,1)$, so the distance power is a positive mixture of Gaussian-kernel deficits. This gives a sign certificate on every Euclidean dimension and every finite point set.