Source task and conventions

Answer both rank and nonuniqueness questions with the source convention $X \in \mathbb{R}^{N\times m}$.

Conventional solution and proof

(a) $XX^\top$ is invertible exactly when X has row rank N, since $v^\top XX^\top v = \|X^\top v\|^2$ vanishes exactly on $\ker X^\top$. This requires $m \geq N$. (b) Let $X = U_r\Sigma_r V_r^\top$ be a compact SVD. Then

$$w_* = (XX^\top)^\dagger Xy = U_r\Sigma_r^{-1}V_r^\top y, \quad X^\top w_* = V_r V_r^\top y = X^\dagger Xy.$$

Consequently $X^\top w = X^\dagger Xy$ holds exactly when $X^\top(w - w_*) = 0$. All solutions are

$$\boxed{w = w_* + v, \quad v \in \ker X^\top} \quad \text{or} \quad w = w_* + (I - XX^\dagger)z,\ z \in \mathbb{R}^N.$$

The projector formula follows from $XX^\dagger = U_r U_r^\top$. Since $w_* \perp \ker X^\top$, $\|w\|^2 = \|w_*\|^2 + \|v\|^2$, proving that w_* is the unique minimum-norm solution even when the prediction-equivalent coefficient vector is not unique.