Source task and conventions

Answer all three elevator-model questions, separating the requested approximation from exact threshold checks.

Conventional solution and proof

For independent passengers with floor probabilities $p_1, \dots, p_F$, the exact no-collision probability for n passengers is $n!e_n(p_1, \dots, p_F)$, where e_n is the elementary symmetric polynomial. This follows by summing products over ordered distinct floors. The expected number of colliding pairs is $\Lambda = \binom{n}{2} \sum_j p_j^2$. The usual rare-collision approximation is $P(\text{collision}) \approx 1 - e^{-\Lambda}$; it is not an identity for a heavily favoured floor.

(1) For uniform floors, $P(\text{no collision}) = (F)_n/F^n = \prod_{j=0}^{n-1}(1 - j/F)$. Expanding $\log(1-u) = -u + O(u^2)$ gives $\log P = -n(n-1)/(2F) + O(n^3/F^2)$ when n/F is small. Thus the approximate threshold solves $n(n-1) > 2F \log 2$. For $F = 30$, both the approximation and exact product give $n = 7$: collision probabilities at $n = 6, 7$ are 0.41355556 and 0.53084444.

(2) If one floor has probability q and all others have $p = (1-q)/(F-1)$, at most one passenger may choose the favoured floor in a collision-free sample. Hence

$$P_0(n) = (F-1)_n p^n + nq(F-1)_{n-1} p^{n-1},$$

with a falling factorial equal to zero when too many distinct floors are requested. The pair approximation uses $s_2 = q^2 + (1-q)^2/(F-1)$ and the first integer n satisfying $\binom{n}{2} s_2 > \log 2$. For $F = 30$, $q = 1/4, 7/20, 1/2$, the approximate and exact thresholds are respectively $5, 4, 3$. Exact collision probabilities at those thresholds are $0.51637260, 0.51051914, 0.52556480$; at one fewer passenger they are $0.36576044, 0.32480380, 0.25862069$. With