Source task and conventions

Prove the five lower-bound and optimality statements and give a sufficient sample size.

Conventional solution and proof

Let P be the m-toss law for coin A, whose head probability is $p = (1-\varepsilon)/2$, and Q that for coin B, whose head probability is $1-p$. Their prior probabilities are equal. The likelihood ratio for a sequence with N heads is

$$P(s)/Q(s) = \left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{2N-m}.$$

It exceeds one exactly when $N < m/2$. Selecting the larger posterior at each sequence minimizes its contribution to the total error; summing proves (e), including arbitrary tie choices at $N = m/2$.

(a) For even m, symmetry gives

$$E_m = P_A(N > m/2) + \tfrac{1}{2}P_A(N = m/2) \geq \tfrac{1}{2}P_A(N \geq m/2),$$

which is the requested inequality.

(b) A complete affinity bound gives a slightly stronger result. Let $\rho = \sum_s \sqrt{P(s)Q(s)} = (\sqrt{1-\varepsilon^2})^m$. By Cauchy–Schwarz,

$$\mathrm{TV}(P,Q) = \tfrac{1}{2}\sum_s |\sqrt{P} - \sqrt{Q}|(\sqrt{P} + \sqrt{Q}) \leq \sqrt{1-\rho^2}.$$

The Bayes error is $E_m = (1 - \mathrm{TV})/2$. Since $\log(1-u) > -u/(1-u)$ for $0 < u < 1$ (differentiate their difference, which vanishes at zero),

$$E_m \geq \tfrac{1}{2}\{1 - \sqrt{1-(1-\varepsilon^2)^m}\} > \tfrac{1}{4}\{1 - \sqrt{1 - e^{-m\varepsilon^2/(1-\varepsilon^2)}}\}.$$