Source task and conventions

Integrate the specified sub-Gaussian tail and resolve the stated range $c > 0$.

Conventional solution and proof

For a nonnegative random variable, Tonelli's theorem applied to $X^2 = \int_0^\infty \mathbf{1}\{X^2 > t\}\, dt$ gives

$$EX^2 = \int_0^\infty P(X^2 > t)dt \le \int_0^\infty \min(1, ce^{-2mt})dt.$$

For $c \ge 1$, split at $u = (\log c)/(2m) \ge 0$. The integral is $u + ce^{-2mu}/(2m) = (\log c + 1)/(2m)$, proving the intended statement. Equivalently, differentiating $u + ce^{-2mu}/(2m)$ on $u \ge 0$ gives its unique minimizer at that point. For $0 < c < 1$, the minimizer over $u \ge 0$ is $u = 0$, and the valid bound is $c/(2m)$. The printed formula for all $c > 0$ is false. Let $X = 0$ with probability $1 - c$ and, with probability c, let $X = \sqrt{Y}$ for $Y \sim \mathrm{Exp}(2m)$. Then $P(X > t) = ce^{-2mt^2}$ for $t > 0$ and $EX^2 = c/(2m) > (1 + \log c)/(2m)$. Thus the sharp consequence of the given tail envelope is

$$\boxed{EX^2 \le \begin{cases} c/(2m), & 0 < c \le 1, \\ (1 + \log c)/(2m), & c \ge 1. \end{cases}}$$

Both branches are attainable by a distribution whose survival function is the displayed minimum.

Structural or analytic derivation

The probability cap $P \le 1$ is a constraint in the scalar optimization. Retaining it determines the breakpoint $c = 1$ and supplies an extremal distribution, which proves optimality of the bound within the stated tail class.