Conventional solution and proof

Let $\phi_i = K(\cdot, x_i)$, $\bar{\phi} = N^{-1}\sum_i \phi_i$, and define the empirical covariance operator

$$Tg = \frac{1}{N}\sum_i \langle g, \phi_i - \bar{\phi}\rangle(\phi_i - \bar{\phi}).$$

It is self-adjoint, positive, finite rank, and $\mathrm{Var}_T\, g(X) = \langle g, Tg\rangle$. Components orthogonal to the centred feature span contribute no variance. On that span the nonzero eigenvalues of T are d_m^2/N, where $JKJ = UD^2U^\top$. Define

$$g_m = \frac{1}{d_m}\sum_i u_{im}(\phi_i - \bar{\phi}) = \frac{1}{d_m}\sum_i u_{im}\phi_i.$$

The second equality holds because positive-eigenvalue u_m is orthogonal to $\mathbf{1}$. Direct inner products give $\langle g_l, g_m\rangle = \delta_{lm}$ and $Tg_m = (d_m^2/N)g_m$. For any unit $g = \sum_m a_m g_m + g_\perp$, Pythagoras gives $\sum_m a_m^2 \le 1$ and

$$\langle g, Tg\rangle = \frac{1}{N}\sum_m d_m^2 a_m^2 \le d_1^2/N.$$

Equality is attained by g_1, proving $\widehat{c}_i = u_{i1}/d_1$. Under orthogonality to $g_1, \ldots, g_{m-1}$, the same calculation bounds the variance by d_m^2/N and gives g_m. Repeated eigenvalues give nonunique bases; when all centred eigenvalues vanish, the variance is zero for every unit function and the displayed division formula has no nonzero component to represent.

Structural or analytic derivation

The coefficient calculation can also be written as maximizing $c^\top K_c^2 c/N$ subject to $c^\top K_c c = 1$, after discarding null directions. Diagonalizing K_c reduces this to a weighted average of its eigenvalues. The operator proof additionally explains the full RKHS and the unobserved orthogonal complement.