Source task and conventions

Prove all four RKHS identities and the equality condition.

Conventional solution and proof

Use the source expansion $K(x,y) = \sum_{j:\gamma_j>0} \gamma_j \phi_j(x)\phi_j(y)$, and write $f = \sum_j c_j \phi_j$ with $\|f\|_H^2 = \sum_j c_j^2/\gamma_j < \infty$. Zero-eigenvalue terms are omitted. At each evaluation point assume the stated RKHS kernel is finite, $K(x,x) < \infty$. Cauchy–Schwarz gives

$$\sum_j |c_j \phi_j(x)| \le \left(\sum_j c_j^2/\gamma_j\right)^{1/2} \left(\sum_j \gamma_j \phi_j(x)^2\right)^{1/2} < \infty.$$

Thus the evaluation and the following inner products are justified. (a) The coefficients of $K(\cdot, x_i)$ are $\gamma_j \phi_j(x_i)$, giving $\langle K(\cdot,x_i), f\rangle_H = \sum_j c_j \phi_j(x_i) = f(x_i)$. (b) Apply (a) to $f = K(\cdot, x_j)$. (c) Bilinearity for the finite expansion $g = \sum_i \alpha_i K(\cdot, x_i)$ gives $J(g) = \|g\|_H^2 = \sum_{i,j} \alpha_i \alpha_j K(x_i, x_j)$. (d) Orthogonality implies $\rho(x_i) = 0$ by (a), and $\langle g, \rho\rangle = 0$. Hence all loss terms are unchanged and $J(g+\rho) = J(g) + \|\rho\|^2$. The difference of objectives is exactly $\lambda\|\rho\|^2$. For $\lambda > 0$ it is nonnegative, with equality exactly when $\rho = 0$. At $\lambda = 0$ the source's exclusive equality condition would fail.