Source task and conventions

Prove the equivalence of optimization problems (18.16)–(18.17), including existence and uniqueness qualifications.

Conventional solution and proof

Assume $X = RV^\top$, $V^\top V = I$, and $\lambda > 0$. Every β has a unique decomposition $\beta = V\theta + \eta$, $V^\top \eta = 0$. Then $X\beta = R\theta$ and $\|\beta\|^2 = \|\theta\|^2 + \|\eta\|^2$. The full objective is therefore exactly

$$\sum_i L(y_i, \beta_0 + r_i^\top \theta) + \lambda\|\theta\|^2 + \lambda\|\eta\|^2.$$

For every fixed (β_0, θ) it is uniquely minimized over η at zero. Thus its infimum equals the reduced problem's infimum, and its attained minimizers correspond bijectively to the reduced minimizers by $(\beta_0, \beta) = (\theta_0, V\theta)$. This proves the theorem as a statement about minimizer sets without assuming differentiability or convexity of L. If no minimum is attained, the equality of infima still holds but a hatted minimizer is not defined. If the reduced loss is not strictly convex in the unpenalized intercept, that intercept need not be unique. At $\lambda = 0$, orthogonal components may be arbitrary; the lifted reduced solution is a minimum-norm representative.