Source task and conventions

Prove (a)–(c), including denominator conditions and the exact deletion-consistency criterion.

Conventional solution and proof

(a) For a fixed penalized least-squares rule, augment a deleted-data minimizer by observation $(x_i, \widehat{f}^{(-i)}(x_i))$. The added squared residual is zero at that minimizer and nonnegative elsewhere, so it remains a minimizer. Under uniqueness, fitted values are identical. Because the full rule is linear, replacing y_i by $t = \widehat{f}^{(-i)}(x_i)$ gives $t = \widehat{f}(x_i) + S_{ii}(t - y_i)$. If $1 - S_{ii} \neq 0$, rearrangement yields the claimed identity. Least-squares projections and fixed-parameter cubic smoothing splines satisfy this self-consistency property, provided the deleted fit is well defined and the unnormalised penalty remains fixed.

(b) For a projection, $S = S^\top = S^2$, so $0 \leq S_{ii} \leq 1$. For a smoothing spline, the fitted-value problem can be represented by minimizing $\|y - f\|^2 + \lambda f^\top \Omega f$ over the feasible evaluation space, with $\Omega \succeq 0$; its smoother