Source task and conventions

Invert the stated rational function in each of its three regions, using residues.

Conventional solution and proof

On an admissible counterclockwise circle, $h[n] = (2\pi i)^{-1} \oint H(z) z^{n-1}\, dz$. The residues at two and three are -2^n and $2\,3^n$. For $n \geq 0$ there is no pole at zero, so the enclosed nonzero poles give respectively both, only the first, or neither contribution. For $n \leq -1$, the integrand is $O(z^{-2})$ at infinity; its residues at zero, two and three sum to zero. Thus the origin residue equals $2^n - 2\,3^n$. Adding whichever nonzero poles lie inside the circle gives zero in the outer region, $-2\,3^n$ in the annulus, and $2^n - 2\,3^n$ in the inner region. This proves the same table by residues for every integer n.

Structural or analytic derivation

For $H(z) = z(z-1)/[(z-2)(z-3)]$, the partial fractions are $-z/(z-2) + 2z/(z-3)$. The complete inverse sequences are

$$\begin{array}{c|l} |z| > 3 & [-2^n + 2\,3^n]u[n] \\ 2 < |z| < 3 & -2^n u[n] - 2\,3^n u[-n-1] \\ |z| < 2 & [2^n - 2\,3^n]u[-n-1]. \end{array}$$

The identities $z/(z-a) = \sum_{n \geq 0} a^n z^{-n}$ for $|z| > |a|$ and $z/(z-a) = -\sum_{n \leq -1} a^n z^{-n}$ for $|z| < |a|$ follow by geometric summation, proving the table and every support endpoint.

Comparative cost and preferred method