## Conventional solution and proof

Since $x[n]$ is real,

$$X^*[(-k)_N] = \sum_{n=0}^{N-1} x[n] e^{2\pi i (-k)_N n/N} = \sum_n x[n] e^{-2\pi i k n/N} = X[k].$$

Writing $X[k] = a_k + ib_k$ then gives $a_{(-k)_N} = a_k$ and $b_{(-k)_N} = -b_k$. In particular $X[0]$ is real, and if $N$ is even then $X[N/2]$ is real. These self-paired cases are part of the proof, not extra assumptions.

## Structural or analytic derivation

The DFT sends real vectors to the fixed-point subspace of circular conjugate reversal. This subspace has exactly $N$ real degrees of freedom, so storing $N$ unrestricted complex numbers is redundant.

## Comparative cost and preferred method

The symmetry reconstructs a full spectrum from $\lfloor N/2 \rfloor + 1$ complex-grid values with the endpoint reality constraints. Completion is linear in output size. A real FFT exploits this redundancy internally; comparison with a strong real-FFT library does not yield another factor-of-two saving.