Source task and conventions

Prove linear phase for $x = (1, 2, 3, 2, 1)$ supported on $0, \dots, 4$.

Conventional solution and proof

Pair terms symmetrically around two:

$$X(e^{i\omega}) = e^{-2i\omega}[3 + 4\cos\omega + 2\cos 2\omega] = e^{-2i\omega}(1 + 2\cos\omega)^2.$$

The bracket is real and nonnegative. Therefore the phase is -2ω modulo 2π wherever $X \neq 0$, and the group delay is two samples. At $\omega = \pm 2\pi/3$ modulo 2π, the response is zero and phase is undefined. The identity also follows by recognizing $(1 + z^{-1} + z^{-2})^2$.

Structural or analytic derivation

Self-reciprocity $z^{-4}X(z^{-1}) = X(z)$ forces the centered response to be real on the circle. The explicit square additionally removes sign-induced π phase jumps. This is an exact polynomial factorization, not a visual inference from a phase plot.

Comparative cost and preferred method

Symmetric evaluation uses two cosines and three real coefficients instead of five complex exponential terms per frequency. The factorized formula uses one cosine, one affine expression and one square. For implementation, coefficient symmetry pairs input samples and reduces distinct nontrivial multipliers; the exact gain depends on treatment of the integer coefficients.