Conventional solution and proof

If $h \in \ell^1(\mathbb{Z})$ and $\|x\|_\infty \le B$, absolute convergence and the triangle inequality give $|(h * x)[n]| \le B \sum_k |h[k]|$ for every n. Conversely, assume the convolution operator obeys a uniform BIBO bound $\|Tx\|_\infty \le C\|x\|_\infty$. For each finite set F, choose a bounded input supported at $-F$ with $x[-k] = \overline{h[k]}/|h[k]|$ for $h[k] \neq 0$ and zero otherwise. Then $(Tx)[0] = \sum_{k \in F} |h[k]| \le C$. Increasing F through finite intervals proves $\sum_k |h[k]| \le C$. This also proves

$$\boxed{\|T\|_{\ell^\infty \to \ell^\infty} = \|h\|_1.}$$

If stability is phrased only as every bounded input producing bounded output, the single input $x[-k] = \overline{h[k]}/|h[k]|$ makes the output at zero diverge whenever $\sum |h[k]| = \infty$, so the convolution is not defined for all bounded inputs.

Structural or analytic derivation

This is the exact functional-analytic operator norm. For a rational causal filter, poles strictly inside the unit disk imply exponentially decaying polynomial-weighted impulse coefficients, hence ℓ^1 summability. A surviving pole on or outside the disk prevents summability. Pole cancellation must be performed on the actual input-output transfer function; hidden internal-state stability is a separate question.

Comparative cost and preferred method

Checking finitely many sample responses cannot prove the universal quantifier over bounded inputs. The ℓ^1 criterion proves it once and supplies the best bound. For a finite impulse response of length M, exactly M absolute values and $M - 1$ additions evaluate that bound; inspecting all coefficients is necessary in the unconstrained input model.